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THE

AMERICAN

MATHEMATICAL MONTHLY

VOLUME XXI SEPTEMBER, 1914 NUMBER 7

# THE APPARENT SIZE OF A CUBE.

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1. The apparent size of any surface is defined as the solid angle of a cone whose vertex is at the point of sight and whose directrix is the curve bounding the surface. The solid angle, being measured by the area cut by the cone from a unit sphere whose center is at the point of sight, may vary from zero to $4\pi$, the apparent size of the whole " sky." The apparent size of a surface as seen from a certain point of space is evidently a function of the position of the point of sight. Its value is given by the integral

$$\omega = \int \frac{\cos (n, r)}{r^2} dS \tag{1}$$

extended over the whole surface, where $r$ is the radius vector from the point of sight to any point of the surface, $dS$ any element of surface, and $(n, r)$ the angle between $r$ and the positive direction of the normal to the surface.

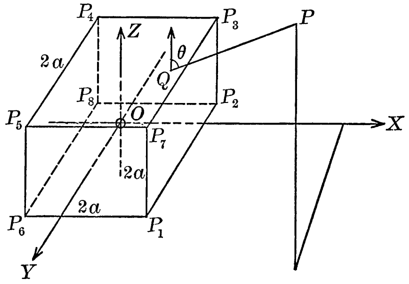

FIG. 1.

We shall take the center of the cube as origin and let the coördinate axes be parallel to the three edges of the cube. Let us consider first the solid angle subtended by the upper face of the cube, i. e., by the square $P_7P_3P_4P_5$. Let

$Q \equiv (\xi, \eta, a)$ be any point in this square and $P \equiv (x, y, z)$ any point in space. Then if $z > a$ the solid angle subtended at $P$ by the square is given by formula (1) where $r = PQ$ and $\cos (n,. r) \equiv \cos \theta = (z - a)/r$, and $dS = d\xi d\eta$.

Hence

$$\omega_1 = \int\int \frac{z-a}{r^3}\, d\xi d\eta = (z-a) \int_{-a}^{+a} \int_{-a}^{+a} \frac{d\xi d\eta}{[(x-\xi)^2 + (y-\eta)^2 + (z-a)^2]^{\frac{3}{2}}}.$$

After performing the integration we have

$$\omega_1 = \arctan \frac{(x-a)(y-a)}{(z-a)\Delta_7} + \arctan \frac{(x+a)(y+a)}{(z-a)\Delta_4} - \arctan \frac{(x+a)(y-a)}{(z-a)\Delta_5} - \arctan \frac{(x-a)(y+a)}{(z-a)\Delta_3}.$$

Likewise the solid angle subtended by the square $P_1P_2P_3P_7$ is given by

$$\omega_2 = \arctan \frac{(y-a)(z-a)}{(x-a)\Delta_7} + \arctan \frac{(y+a)(z+a)}{(x-a)\Delta_2} - \arctan \frac{(y+a)(z-a)}{(x-a)\Delta_3} - \arctan \frac{(y-a)(z+a)}{(x-a)\Delta_1};$$

and for the solid angle subtended by the square $P_5P_6P_1P_7$ we obtain

$$\omega_3 = \arctan \frac{(z-a)(x-a)}{(y-a)\Delta_7} + \arctan \frac{(z+a)(x+a)}{(y-a)\Delta_6} - \arctan \frac{(z+a)(x-a)}{(y-a)\Delta_1} - \arctan \frac{(z-a)(x+a)}{(y-a)\Delta_5},$$

where

$$\begin{aligned} \Delta_1^2 &= (x-a)^2 + (y-a)^2 + (z+a)^2, & \Delta_4^2 &= (x+a)^2 + (y+a)^2 + (z-a)^2, \\ \Delta_2^2 &= (x-a)^2 + (y+a)^2 + (z+a)^2, & \Delta_5^2 &= (x+a)^2 + (y-a)^2 + (z-a)^2, \\ \Delta_3^2 &= (x-a)^2 + (y+a)^2 + (z-a)^2, & \Delta_6^2 &= (x+a)^2 + (y-a)^2 + (z+a)^2, \end{aligned} \tag{2}$$

$$\Delta_7^2 = (x-a)^2 + (y-a)^2 + (z-a)^2.$$

2. The solid angle subtended by the entire cube will be the sum of one, two, or three of these solid angles according as one, two, or three faces of the cube are visible. Hence we must consider three different cases.

*Case A.* $\mathbf{x > a,\ -a < y < a,\ -a < z < a}$. In this case only one face is visible and $\omega = \omega_2$. Now by means of the formulas

$$\arctan(-m) = -\arctan(m) \quad \text{and} \quad \arctan(m) = \pi/2 - \arctan(1/m)$$

we can write the value of $\omega$ in the form

$$\omega = 2\pi - \left[\arctan\frac{(x-a)\Delta_1}{(a-y)(a+z)} + \arctan\frac{(x-a)\Delta_2}{(a+y)(a+z)} + \arctan\frac{(x-a)\Delta_3}{(a+y)(a-z)} + \arctan\frac{(x-a)\Delta_7}{(a-y)(a-z)}\right]. \tag{3}$$

*Case B.* $x > a$, $y > a$, $-a < z < a$. In this case two faces are visible and $\omega = \omega_2 + \omega_3$. If we combine the terms that contain the same radical by means of the formula

$$\arctan(m) + \arctan(n) = \arctan\frac{m+n}{1-mn}$$

we obtain, after further reduction,

$$\omega = 2\pi - \left[\arctan\frac{(a+z)\Delta_1}{(x-a)(y-a)} + \arctan\frac{(x-a)\Delta_2}{(y+a)(a+z)} + \arctan\frac{(x-a)\Delta_3}{(y+a)(a-z)} + \arctan\frac{(a-z)\Delta_7}{(x-a)(y-a)} + \arctan\frac{(y-a)\Delta_5}{(a-z)(x+a)} + \arctan\frac{(y-a)\Delta_6}{(x+a)(a+z)}\right]. \tag{4}$$

*Case C.* $x > a$, $y > a$, $z > a$. In this case three faces are visible and $\omega = \omega_1 + \omega_2 + \omega_3$. After combining terms as before we obtain

$$\omega = 2\pi - \left[\arctan\frac{(z+a)\Delta_1}{(x-a)(y-a)} + \arctan\frac{(x+a)\Delta_5}{(y-a)(z-a)} + \arctan\frac{(y+a)\Delta_3}{(z-a)(x-a)} + \arctan\frac{(z-a)\Delta_4}{(x+a)(y+a)} + \arctan\frac{(x-a)\Delta_2}{(y+a)(z+a)} + \arctan\frac{(y-a)\Delta_6}{(z+a)(x+a)}\right]. \tag{5}$$

It is not necessary to consider any more than these three cases since, for any given value of $\omega$, the surfaces given by equations (3), (4), and (5) will be symmetrical with respect to the coördinate planes and the origin. Hence it is sufficient to study the formulas for one octant with different regions according to the number of faces visible. If the upper face alone is visible we can obtain the value of the solid angle by simply interchanging $x$ and $z$ in equation (3); if faces $P_1P_2P_3P_7$ and $P_2P_3P_4P_8$ are visible, simply change $y$ to $-y$ in (4), etc.

It may be interesting to note that from any point of sight the cube will project into either a quadrilateral or a hexagon and in each of the equations (3), (4), (5) the radicals, which we have denoted by $\Delta$, represent the distances from the point of sight to the vertices of that quadrilateral or hexagon.

3. The solid angle subtended by the cube may also be obtained from the relation $\omega = 2\pi - \sigma$ where $\sigma$ is the length of that curve on the unit sphere which is polar to the intersection of the cone and the unit sphere. In this problem $\sigma$ is the sum of the dihedral angles of the cone. Of course this cone is really a pyramid since its directrix is made up of straight lines.

If in each of the three cases we write the equations of the planes forming the lateral faces of the pyramids and then find the value of $\sigma$ for each pyramid, the formula $\omega = 2\pi - \sigma$ leads to equations (3), (4), (5).

4. It may be interesting to note the behavior of these formulas as the point $P$ moves away from the cube in any direction. If we change to polar coördinates and then express $\omega_1$ as a power series in $1/r$, it will be found that the series begins with the term in $1/r^2$. In fact

$$\omega_1 = 4a^2 \cos (n, r) \times 1/r^2 + \cdots \text{ higher powers of } 1/r,$$

where $(n, r)$ is the angle that the radius vector makes with the normal to the face. It will be found that $\omega_2$ and $\omega_3$ may be written in the same form. Hence we see that in each of the three cases (when one, two or three faces are visible) the apparent size of the cube will approach zero as the point of sight approaches infinity in any direction; and also that $\omega r^2$ will approach

$$\Sigma(\text{area of face}) \times (\text{cosine of the angle between the radius vector and the normal to the face}).$$

In other words, as the point of sight approaches infinity in any direction, each of the faces of the cube acts more and more as if it were a single differential element $\cos\theta \, dS/r^2$.

5. If we combine certain terms in each of the equations (3), (4), (5) we are led to the following results: (*a*) when only one face is visible the solid angle may have any value between zero and $2\pi$, being equal to $2\pi$ only when the point of sight lies on the face of the cube; (*b*) when only two faces are visible the solid angle lies between zero and $\pi$, being equal to $\pi$ only when the point of sight lies on one edge of the cube; (*c*) when three faces are visible the solid angle lies between zero and $\pi/2$, being equal to $\pi/2$ only when the point of sight is at one corner of the cube.

If we let $\omega$ be constant in equations (3), (4), (5) we have in each case the equation of a surface. The apparent size of the cube will not vary as long as the point of sight remains on this surface. We propose now to study the forms of these surfaces for different values of $\omega$.

6. Let us assume all space outside the cube to be divided into the following regions: (*a*) six prisms bounded by the six faces of the cube produced and extending, one from each face of the cube, to infinity; (*b*) twelve wedge-shaped regions extending, one from each edge, to infinity and bounded by the two faces which are perpendicular to this edge and the two faces which pass through the edge; (*c*) eight three-edged regions extending, one from each corner of the cube, to infinity and bounded by the three faces produced which meet in the given corner.

Then if $\omega$ has a constant value, such that all three of our surfaces exist, we have:

*Case A.* six distinct surfaces, one in each of the regions ($a$), symmetrical in pairs with respect to the three lines joining the center of the cube to the mid-points of the faces and symmetrical in pairs with respect to the origin;

*Case B.* twelve distinct surfaces, one in each of the regions ($b$), symmetrical in pairs with respect to the lines joining the mid-points of opposite edges of the cube and symmetrical in pairs with respect to the origin;

*Case C.* eight distinct surfaces, one in each of the regions ($c$), symmetrical in pairs with respect to the diagonals of the cube and symmetrical in pairs with respect to the origin.

It is not necessary to study each of these twenty-six different surfaces since any one of them may be obtained from one of the equations (3), (4), (5) by various permutations of letters and signs. For simplicity we shall refer to the surface from which only one face is visible as surface I or $F_1(x, y, z) = 0$; the surface from which two faces are visible we shall call surface II or $F_2(x, y, z) = 0$; and the surface from which three faces are visible we shall call surface III or $F_3(x, y, z) = 0$. Their equations are given by (3), (4), (5).

7. We propose now to obtain the equations of the curves in which the planes of the faces produced intersect surfaces I, II, and III. Of course it must be understood in all that follows that $\omega < \pi/2$, otherwise some of the surfaces may not exist. The plane $y = a$ separates the region of space in which surface I lies from that in which surface II is found. It can be shown easily that both these surfaces are cut by the plane $y = a$ in the same curve, whose equation is

$$\omega = \pi - \left[\arctan\frac{(x-a)\Delta_2}{2a(a+z)} + \arctan\frac{(x-a)\Delta_3}{2a(a-z)}\right]. \tag{6}$$

Likewise surfaces II and III are both cut by the plane $z = a$ in the same curve, whose equation is

$$\omega = \pi - \left[\arctan\frac{2a\Delta_1}{(x-a)(y-a)} + \arctan\frac{(x-a)\Delta_2}{2a(y+a)} + \arctan\frac{(y-a)\Delta_6}{2a(x+a)}\right]. \tag{7}$$

Hence we see that although we have twenty-six different surfaces defined by twenty-six distinct equations, they come together in pairs on the faces of the cube produced and form a single surface completely surrounding the cube and continuous everywhere.

8. Let us see whether any pair of these surfaces have a common tangent plane at any point of their curve of intersection. The direction cosines of the normal to the tangent plane to any surface are proportional to $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$, where $F(x, y, z) = 0$ is the equation of the surface. These partial derivatives may be easily obtained from equations (3), (4), (5), and it is not necessary to write their values here.

Since surfaces I and II meet on the plane $y = a$ we must investigate the values of $\partial F_1/\partial x$, $\partial F_1/\partial y$, $\partial F_1/\partial z$, $\partial F_2/\partial x$, $\partial F_2/\partial y$, $\partial F_2/\partial z$ for $y = a$. If at any point on the curve of intersection we find the first three of these derivatives respectively equal to or proportional to the last three then the two surfaces have a common tangent plane at that point. It can be easily shown that $\partial F_1/\partial x \equiv \partial F_2/\partial x$ and $\partial F_1/\partial z \equiv \partial F_2/\partial z$, and $\partial F_1/\partial y \not\equiv \partial F_2/\partial y$. Hence the two surfaces do not have a common tangent plane at *every* point of their curve of intersection. Let us see whether they have a common tangent plane at *any* point of their curve of intersection. If $\partial F_1/\partial y = \partial F_2/\partial y$ it easily follows that

$$(x + a)(a - z)\Delta_1 - (x - a)(a - z)\Delta_6 = (x - a)(a + z)\Delta_5 - (x + a)(a + z)\Delta_7.$$

After a rather lengthy reduction it can be shown that this equation is not possible unless $x = a$ and $z = \pm a$, i. e., unless the curve of intersection passes through one corner of the cube. This cannot happen except for $\omega = \pi/2$, which will be considered later. Hence, if $\omega < \pi/2$, the surfaces I and II have no common tangent plane at any point of their curve of intersection.

Likewise it can be shown that, if $\omega < \pi/2$, surfaces II and III have no common tangent plane at any point of their curve of intersection.

9. We have said that each of the surfaces I, II, and III is symmetrical with respect to a certain line which we shall call the axis of the surface. The $X$-axis is the axis of surface I; the line joining the origin to the point $(a, a, 0)$ is the axis of surface II; and surface III has for its axis the line joining the origin to the point $(a, a, a)$, the corner of the cube. We propose to show that each axis is a normal to its surface.

The axis of surface I will cut it at some point on the $X$-axis. Hence we must investigate the values of $\partial F_1/dx$, $\partial F_1/\partial y$, $\partial F_1/\partial z$ when $y = z = 0$. We find

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_7, \quad \text{and} \quad \frac{\partial F_1}{\partial x} = \frac{-8a^2}{\Delta_1[(x-a)^2 + a^2]}, \quad \frac{\partial F_1}{\partial y} = 0, \quad \frac{\partial F_1}{\partial z} = 0.$$

Hence the direction cosines of the normal at the point $(x, 0, 0)$ are 1, 0, 0. In other words the $X$-axis is the normal.

The axis of surface II will cut the surface at some point on the line $y = x$, $z = 0$. If we put $y = x$ and $z = 0$ we find $\Delta_1 = \Delta_7$, $\Delta_2 = \Delta_3 = \Delta_5 = \Delta_6$, and $\partial F_2/\partial x = \partial F_2/\partial y \neq 0$, and $\partial F_2/\partial z = 0$. Hence the direction cosines of the normal at this point are proportional to 1, 1, 0; i. e. $\lambda = 1/\sqrt{2}$, $\mu = 1/\sqrt{2}$, $\nu = 0$. Hence the axis is the normal.

The axis of surface III will cut the surface at some point where $z = y = x$. If we put $z = y = x$ we find $\Delta_1 = \Delta_3 = \Delta_5$ and $\Delta_2 = \Delta_4 = \Delta_6$. Also $\partial F_3/\partial x = \partial F_3/\partial y = \partial F_3/\partial z$. Hence the direction cosines of the normal are $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$. In other words the line $y = z = x$ is the normal.

10. In order to make the surfaces more vivid geometrically we shall find the coördinates of several points on them. Let us first find the length of the intercept of each surface on its axis.

Surface I. Put $z = y = 0$ in equation (3) and obtain

$$\frac{\pi}{2} - \frac{\omega}{4} = \arctan \frac{\sqrt{(x-a)^2 + 2a^2}}{a^2}.$$

Solving for $x$ we have $x = a[1 \pm \sqrt{\csc(\omega/4) - 1}]$. We must take the positive sign in order to be on the positive side of the face. Hence

$$x = a\left[1 + \sqrt{\csc \frac{\omega}{4} - 1}\right]. \tag{8}$$

This gives us the length of $OA_z = OA_y = OA_x$. (Fig. 2.)

Surface II. Putting $z = 0$ and $y = x$ in equation (4), we obtain

$$\omega = 2\pi - \left[2 \arctan \frac{a\Delta_1}{(x-a)^2} + 4 \arctan \frac{(x-a)\Delta_2}{a(x+a)}\right], \quad \Delta_1{}^2 = 2x^2 - 4ax + 3a^2,$$

or

$$\frac{\pi}{2} - \frac{\omega}{4} = \frac{1}{2} \arctan \frac{a\Delta_1}{(x-a)^2} + \arctan \frac{(x-a)\Delta_2}{a(x+a)}, \quad \Delta_2{}^2 = 2x^2 + 3a^2.$$

Let

$$\varphi = \arctan \frac{a\Delta_1}{(x-a)^2}; \quad \text{then} \quad \tan \varphi = \frac{a\Delta_1}{(x-a)^2}$$

and

$$\frac{\varphi}{2} = \arctan \frac{1 - \cos \varphi}{\sin \varphi} = \arctan \frac{a}{\Delta_1}.$$

Hence

$$\tan\left(\frac{\pi}{2} - \frac{\omega}{4}\right) = \arctan \frac{a}{\Delta_1} + \arctan \frac{(x-a)\Delta_2}{a(x+a)},$$

then

$$\tan\left(\frac{\pi}{2} - \frac{\omega}{4}\right) = \cot \frac{\omega}{4} = \frac{a^2(x+a) + (x-a)\Delta_1\Delta_2}{a(x+a)\Delta_1 - a(x-a)\Delta_2}.$$

Rationalizing the denominator, we obtain

$$\cot \alpha = \frac{(x^2 + a^2)\Delta_1 + (x^2 - a^2)\Delta_2}{2a^2x}, \quad \text{where} \quad \alpha = \frac{\omega}{4},$$

or

$$2a^2x \cot \alpha - (x^2 - a^2)\Delta_2 = (x^2 + a^2)\Delta_1.$$

If we rationalize this equation we obtain

$$\begin{aligned}(1-c)^2x^8 &- 4(1-c)^2ax^7 + 2(1-c)(3-5c)a^2x^6 - 4(1-c)(3-4c)a^3x^5 \\ &+ 3(1-c)(5-7c)a^4x^4 - 4(1-c)(3-5c)a^5x^3 + 4(3-6c+4c^2)a^6x^2 \\ &- 4(1-c)(1-2c)a^7x - 2(1-c)(1+2c)a^8 = 0,\end{aligned} \tag{9}$$

where $c$ is written in place of $\cos 2\alpha = \cos(\omega/2)$ for simplicity. This equation has at least one positive root greater than $a$ and this root can be found after $c$ is known. It may happen for certain values of $c$ that the equation has more than one positive root greater than $a$, in which case we must choose the root which satisfies the equation

$$\cot\frac{\omega}{4} = \frac{(x^2+a^2)\Delta_1 + (x^2-a^2)\Delta_2}{2a^2x}.$$

If $x_1$ is the required root of equation (9) then the length of the intercept is $x_1\sqrt{2}$. This gives us the length of $OB_z = OB_y = OB_x$ in Fig. 2.

Surface III. Putting $z = y = x$ in equation (5), we obtain

$$\omega = 2\pi - \left[3\arctan\frac{(x+a)\Delta_1}{(x-a)^2} + 3\arctan\frac{(x-a)\Delta_2}{(x+a)^2}\right]$$

where

$$\Delta_1^2 = 2(x-a)^2 + (x+a)^2;\quad \Delta_2^2 = 2(x+a)^2 + (x-a)^2.$$

From this equation we obtain, after a rather lengthy reduction,

$$x = \pm a\sqrt{\tan\left(\frac{\pi}{6} - \frac{\omega}{12}\right)\cot\frac{\omega}{4}}. \tag{10}$$

Hence $OC = x\sqrt{3} = a\sqrt{3\tan(\pi/6 - \omega/12)\cot(\omega/4)}$.

In the case of surfaces I and III we have found the length of the intercept as an explicit function of $\omega$, but in the case of surface II we did not obtain a simple result. The reason for this is obvious geometrically. If we are considering only one face of the cube, surface I is symmetrical with respect to the plane of this face and hence we obtain $x - a = \pm f(\omega)$. Likewise when we are considering three faces the surface is symmetrical with respect to the origin and we obtain $x = \pm\varphi(\omega)$. This symmetry does not exist in the case of surface II. The solid angle will not be the same for two points on the axis such that $x = \pm\psi(\omega)$ if we are looking at the *same* two faces. That is, equation (9) should not have two roots of the form $x = \pm\psi(\omega)$. It can be shown easily that equations (8), (9), and (10) hold for limiting values of $\omega$. When $\omega = 0$, each equation gives $x = \infty$; when $x = a$ we have $\omega = 2\pi$, $\pi$, and $\pi/2$ respectively.

11. We have seen that surfaces I and II are both cut by the plane $y = a$ in the curve

$$\pi - \omega = \arctan\frac{(x-a)\Delta_2}{2a(a+z)} + \arctan\frac{(x-a)\Delta_3}{2a(a-z)}. \tag{11}$$

The arc $Q_1M_z$ (Fig. 2) represents the upper half of this curve. If we find $dz/dx$ from (11) it can be easily shown that the curve $Q_1M_z$ cuts the $XY$ plane at right angles for every value of $\omega$, and that it makes with the edge $SP_7$ (produced) an acute angle whose value depends upon $\omega$. In fact this angle varies from zero to $\pi/2$ as $\omega$ varies from $\pi/2$ to zero.

Putting $z = 0$ in (11), we obtain $\pi - \omega = 2\arctan(x-a)\Delta_2/2a^2$, whence

$$\cot(\omega/2) = \tan(\pi/2 - \omega/2) = (x-a)\Delta_2/2a^2.$$

Then

$$\cot^2(\omega/2) = (x-a)^2[(x-a)^2 + 5a^2]/4a^4$$

or

$$(x-a)^4 + 5a^2(x-a)^2 - 4a^4\cot^2(\omega/2) = 0,$$

whence

$$x - a = HM_z = a\sqrt{\frac{-5 + \sqrt{25 + 16\cot^2(\omega/2)}}{2}}. \tag{12}$$

Hence, when $\omega$ is known, we have the lengths of the lines $HM_z = HM_z' = LM_y = LM_y' = SM_x = SM_x'$.

Now putting $z = a$ in (11), we obtain after reduction

$$x - a = Q_1P_7 = 2a\sqrt{\csc\omega - 1}. \tag{13}$$

Hence we have the lengths of the lines $Q_1P_7 = Q_2P_7 = Q_3P_7$.

12. Let us now study the curve of intersection of surfaces II and III. This is the curve $Q_1Q_2$ (Fig. 2). Its equation is

$$\pi - \omega = \arctan\frac{2a\Delta_1}{(x-a)(y-a)} + \arctan\frac{(x-a)\Delta_2}{2a(y+a)} + \arctan\frac{(y-a)\Delta_6}{2a(x+a)}. \tag{14}$$

If we find the value of $dy/dx$ from (14) we can easily show that the curve cuts the diagonal (produced) of the upper face at right angles, and that it makes with the positive direction of $P_7Q_1$ an obtuse angle whose size depends on the value of $\omega$.

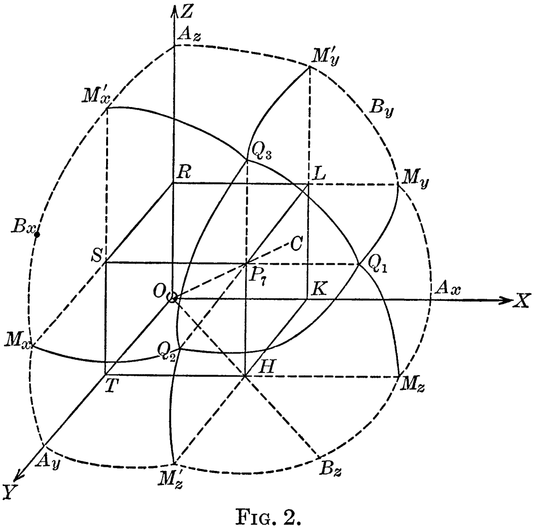

FIG. 2.

In fact as $x$ approaches $a$, $dy/dx$ approaches $-1$; and as $x$ approaches $\infty$, $dy/dx$ approaches $-\infty$. Hence this obtuse angle varies from $3\pi/2$ to $\pi/2$.

We can find the length of $P_7Q_1$ by putting $y = a$ in equation (14). This gves the same result as before.

13. Curvature properties. Let $U=\partial F/\partial x$, $V=\partial F/dy$, $W=\partial F/\partial z$, $u=\partial^2 F/\partial x^2$, $v = \partial^2 F/\partial y^2$, $w = \partial^2 F/\partial z^2$, $u' = \partial^2 F/\partial y \partial z$, $v' = \partial^2 F/\partial z \partial x$, $w' = \partial^2 F/\partial x \partial y$. Also $H = u + U(Uu' - Vv' - Ww')/VW$, $K = v + V(Vv' - Ww' - Uu')/WU$, $L = w + W(Ww' - Uu' - Vv')/UV$. Then the principal radii of curvature at any point are given by the quadratic equation

$$\frac{U^2}{H - Q/\rho} + \frac{V^2}{K - Q/\rho} + \frac{W^2}{L - Q/\rho} = 0, \quad \text{where} \quad Q^2 = U^2 + V^2 + W^2.$$

From the results already obtained it is geometrically reasonably certain that the parts of the surface between the curves of discontinuous slope are everywhere convex outward. This can be proved by showing that for any given point $(x_1, y_1, z_1)$ the two values of $\rho$ obtained from the equation above are both negative.

After finding the values of $\partial^2 F/\partial x^2$, $\partial^2 F/\partial y^2$, $\partial^2 F/\partial z^2$ it can be easily shown that $\partial^2 F/\partial x^2 + \partial^2 F/\partial y^2 + \partial^2 F/\partial z^2 = 0$. That is, the apparent size corresponds to a magnetic potential function.

14. We have assumed up to this point that $\omega$ is less than $\pi/2$. Let us now consider the forms of the different surfaces for other values of $\omega$.

$\omega = \pi/2$. In this case surface III reduces to a mere point, the corner of the cube (See Fig. 3), and we have to study only surfaces I and II. The edge $SP_7$ is now tangent to both surfaces as we have seen before (Art. 8). Putting $\omega = \pi/2$ in (8) we find $OA_x = a(1 + \sqrt{\csc(\pi/8) - 1})$. The equation of the curve of intersection of surfaces I and II now becomes

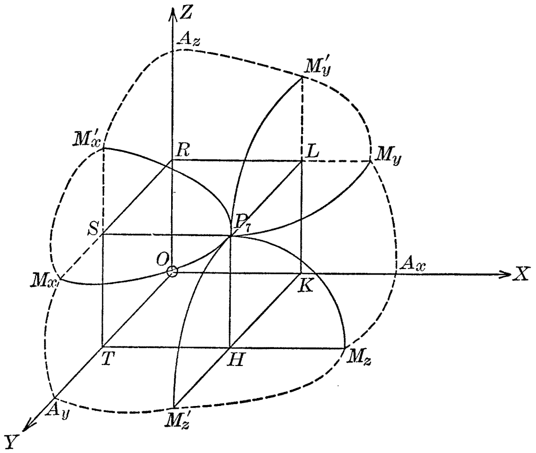

FIG. 3.

$$\frac{\pi}{2} = \arctan\frac{(x - a)\Delta_2}{2a(a + z)} + \arctan\frac{(x - a)\Delta_3}{2a(a - z)}.$$

From this equation we obtain after reduction, $4a^2(a^2 - z^2) = (x - a)^2\Delta_2\Delta_3$. Squaring both sides, we obtain, after reduction,

$$(x - a)^6 + 2(x - a)^4(3a^2 + z^2) + (x - a)^2(a^2 - z^2)^2 - 4a^2(a^2 - z^2)^2 = 0. \quad (15)$$

This is the algebraic equation of the curve of intersection. Putting $z = 0$ we have

$$(x - a)^6 + 6a^2(x - a)^4 + a^4(x - a)^2 - 4a^6 = 0, \quad \text{or} \quad (x-a)^4 + 5a^2(x-a)^2 - 4a^4 = 0,$$

after dividing by $(x - a)^2 + a^2$. This gives

$$HM_z = (x - a) = a\sqrt{\frac{-5 + \sqrt{41}}{2}} = 0.8a,$$

approximately.

If we put $\omega = \pi/2$ in equation (12) we obtain the same result.

Putting $z = a$ in (15) we have $(x - a)^6 + 8a^2(x - a)^4 = 0$, whence $P_7Q_1 = x - a = 0$. This agrees with (13) for $\omega = \pi/2$. It can be easily shown from (15) that $dz/dx = 0$ for $z = x = a$ and $dz/dx = \infty$ for $z = 0$. Hence the point $Q_1$ is at $P_7$ and the curve $P_7M_z$ is tangent to the edge $SP_7$ at $P_7$.

$\pi/2 < \omega < \pi$. In this case surfaces I and II still exist and $OA_x$ is still given by (8) and $HM_z$ by (12). The equation of the curve of intersection of the surfaces I and II, equation (11), is not satisfied by $x = a$ for any value of $z$, nor by $z = a$ for any value of $x$. In other words the curve of intersection approaches the point $P_7$ but never reaches it. The curve is still perpendicular to the $XY$ plane at $M_z$.

$\omega = \pi$. The equation of surface II now reduces to $x = a$, $y = a$, the edge of the cube, and the curve of intersection of surfaces I and II reduces to $x = a$. This is the equation of the edge $P_7P_1$ with the end points left off, since $z$ cannot be equal to $a$. The length of $OA_x$ is now equal to $a[1 + \sqrt{\sqrt{2} - 1}]$.

$\pi < \omega < 2\pi$. In this case surface I alone exists and $OA_x$ is still given by (8). This surface gradually closes down on the face of the cube as $\omega$ approaches $\pi/2$.

15. Forms of different level surfaces. By "level surfaces" we mean surfaces from which the cube has the same apparent size.

*A*. $0 < \omega < \pi/2$. In this case the surface consists of twenty-six distinct parts which form one continuous surface completely surrounding the cube and which approaches a sphere as $\omega$ approaches 0.

*B*. $\omega = \pi/2$. Eight of these surfaces now reduce to mere points, and the remaining eighteen form a continuous surface completely surrounding the cube and touching it at its corners.

*C*. $\pi/2 < \omega < \pi$. Only eighteen surfaces exist and they form a surface continuous everywhere except at the corners of the cube. This is of the same form as case *B* except the corners are left out.

*D*. $\omega = \pi$. Twelve of the surfaces reduce to lines (the edges of the cube), and the remaining six form a surface continuous everywhere except at the corners of the cube and touching the cube along its twelve edges.

*E*. $\pi < \omega < 2\pi$. Only six surfaces exist; one opposite each face. They form a surface completely surrounding the cube except that each edge is a line of discontinuity and each corner is a point of discontinuity.